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Construction of Second Order Slope Rotatable Designs using Partially Balanced Incomplete Block Type Designs

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SUMMARY

A new method of construction of second order slope rotatable designs (SOSRDs) using partially balanced incomplete block (PBIB) type designs is suggested. It is observed that the method sometimes leads to designs with less number of design points than those available in the literature.

Key words : Response surface designs, Slope rotatability, Second order slope rotatable designs.

1. Introduction

A second order response surface design D = ((x_{iu})) for fitting,

$$y_u(x_{1u}, x_{2u}, \ldots, x_{vu}) = b_o + \sum_{i=1}^{v} b_i x_{iu} + \sum\sum_{i \le j} b_{ij} x_{iu} x_{ju} + e_u$$

where x_{iu} denotes the level of i th factor (i=1, 2,...,v) in the u th run (u=1,2,...,N) of the experiment, is said to be a second order slope rotatable design (SOSRD) if D satisfies the following conditions (c.f. Hader and Park [4] and Victorbabu and Narasimham [8]).

A. $\sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0$ if any α_i is odd, for $\alpha_i = 0, 1, 2, 3$ and $\sum \alpha_i \le 4$

B. (i) $\sum_{u=1}^{N} x_{iu}^2 = \text{constant} = N\mu_2$

(ii) $\sum_{u=1}^{N} x_{iu}^4 = \text{constant} = cN\mu_4$

C. $\sum_{u=1}^{N} x_{iu}^2 \, x_{ju}^2 = \text{constant} = N \mu_4$

D. $\frac{\mu_4}{\mu_2^2} = \frac{v(c-5)+4}{(c-3)^2 + v(c-5)}$

E. $\frac{\mu_4}{\mu_2^2} > \frac{v}{c+v-1}$ (non-singularity condition) ...(1.1)

where c, μ_2 and μ_4 are constants and the summation is over the design points.

2. *New Method of Construction of SOSRD using PBIB Type Designs*

Take an incomplete block arrangement with constant block size and replication in which some pairs of treatments occur λ_1 times each ($\lambda_1 \neq 0$) (the design need not be PBIBD). Take this as the first design. For the second design take the incomplete block design with all missing pairs (in the first design) once each with k=2, λ'_1=0, λ'_2=1. Such pairs of PBIB type designs can be constructed in a straight forward manner in particular using existing two-associate PBIB designs with one of the λ's equal to zero.

The method of constructing SOSRD using the above two PBIB type designs is given below. Here we use the notations of Das and Narasimham [2], Narasimham *et al.* [5], Victorbabu and Narasimham [8], [9], [10] and [11].

Let $D_1 = (v, b_1, r_1, k_1, \lambda_1 \neq 0, \lambda_2 = 0)$ be an incomplete block design with constant replication in which only some pair of treatments occur a constant number of times λ_1 ($\lambda_2 = 0$). $[1-(v, b_1, r_1, k_1, \lambda_1, \lambda_2 = 0)]$ denote the design points generated from the transpose of the incidence matrix of incomplete block design.

$[1-(v, b_1, r_1, k_1, \lambda_1, \lambda_2 = 0)]\, 2^{k_1}$ are the $b_1 2^{k_1}$ design points generated from D_1 by "multiplication" (c.f. Raghavarao (1971, pp.298-300)). Let $D_2 = (v, b_2, r_2, k_2 = 2, \lambda_1' = 0, \lambda_2' = 1)$ be the associated second design containing only the missing pairs of treatments of above design D_1. Let $[a_1 - (v, b_2, r_2, k_2 = 2, \lambda_1' = 0, \lambda'_2 = 1)]\, 2^{k_2}$ are the $b_2 2^2$ design points generated from D_2 by multiplication. $(a, 0, \ldots, 0)\, 2^1$ denote the design points generated from (a, 0, ..., 0) point set.

[We use below the $\cup$ symbol to denote combination of the design points generated from the above different sets of points].

Theorem : The design points,

$$[1-(v, b_1, r_1, k_1, \lambda_1, \lambda_2 = 0)]\ 2^{k_1} \cup$$

$$[a_1-(v, b_2, r_2, k_2 = 2, \lambda_1' = 0,\ \lambda_2' = 1)]\ 2^2 \cup (a, 0, \dots, 0)\, 2^1 \cup (n_o)$$

give a v-dimensional SOSRD in $N = b_1\, 2^{k_1} + b_2\, 2^2 + 2v + n_o$ design points.

Proof: For the design points generated from designs D_1 and D_2, conditions (A) of (1.1) are true obviously. Conditions (B) and (C) of (1.1) are true as follows :

$$\sum x_{iu}^2 = r_1\, 2^{k_1} + r_2\, 2^{k_2} a_1^2 + 2a^2 = N\mu_2 \qquad \text{...(1)}$$

$$\sum x_{iu}^4 = r_1\, 2^{k_1} + r_2\, 2^{k_2} a_1^4 + 2a^4 = cN\mu_4 \qquad \text{...(2)}$$

$$\sum x_{iu}^2\, x_{ju}^2 = \lambda_1\, 2^{k_1} \qquad \text{...(3)}$$

$$= \lambda_2'\, 2^{k_2}\, a_1^4 = N\mu_4 \qquad \text{...(4)}$$

From (3) and (4), we get $a_1^4 = \dfrac{\lambda_1}{\lambda_2'}\, 2^{k_1-k_2} = \lambda_1\, 2^{k_1-2}$ ...(5)

(since $\lambda_2' = 1,\ k_2 = 2$)

Using (2), (3) and (4), we get

$$r_1\, 2^{k_1} + r_2 \left(\frac{\lambda_1}{\lambda_2'}\right) 2^{k_1} + 2a^4 = c\lambda_1\, 2^{k_1}$$

$$\Rightarrow \qquad a^4 = 2^{k_1-1}\,(c\lambda_1 - r_1 - r_2\,\lambda_1) \qquad \text{...(6)}$$

Using (1), (3) and (5) and substituting for a_1^2, we get

$$\frac{v\,(c-5)+4}{N\,[(c-3)^2 + v\,(c-5)]} = \frac{\lambda_1\, 2^{k_1}}{[r_1\, 2^{k_1} + r_2\,(\lambda_1)^{1/2}\, 2^{(k_1+2)/2} + 2a^2]^2} \qquad \text{...(7)}$$

putting $c_1 = c-3$ in (6) and (7), we get

$$a^4 = 2^{k_1-1}\,(c_1\,\lambda_1 + 3\lambda_1 - r_1 - r_2\,\lambda_1) \qquad \text{...(8)}$$

$$\frac{v(c_1-2)+4}{N[c_1^2+v(c_1-2)]}=\frac{\lambda_1 2^{k_1}}{[r_1 2^{k_1}+r_2(\lambda_1)^{1/2}2^{(k_1+2)/2}+2a^2]^2} \quad ...(9)$$

Equation (9) in the unknowns c_1 or a^2 can be solved iteratively.

[Alternatively eliminating c_1 (or a^2) from (8) or (9), we get a 4th degree equation in a^2 (or c_1)].

Example (i) : We illustrate the construction of SOSRD in 6-factors using the PBIB type designs,

$$D_1 = (v = 6, b_1 = 4, r_1 = 2, k_1 = 3, \lambda_1 = 1, \lambda_2 = 0)$$

and the associated second design D_2 with missing pairs in D_1.

Here $D_1 = \{(1, 2, 3)\ (1, 5, 6)\ (2, 4, 6)\ (3, 4, 5)\}$

$D_2 = \{(3, 6)\ (2, 5)\ (1, 4)\}$

The design points,

$[1-(v = 6,\ b_1 = 4,\ r_1 = 2,\ k_1 = 3,\ \lambda_1 = 1,\ \lambda_2 = 0)]\ 2^3 \cup$

$[a_1-(v = 6, b_2 = 3,\ r_2 = 1,\ k_2 = 2,\ \lambda_1' = 0,\ \lambda_2' = 1)]\ 2^2 \cup$

$(a, 0, \ldots, 0)\ 2^1 \cup (n_o = 1)$

will give a SOSRD in N=57 design points for six factors.

Here

$$(8) \rightarrow a^4 = 4c_1 \ (\text{as } \lambda_1 = \lambda_2' = 1, r_1 = 2,\ r_2 = 1) \quad ...(10)$$

$$(9) \rightarrow \frac{6c_1-8}{N[c_1^2+6c_1-12]}=\frac{8}{[16+2^{5/2}+2a^2]^2} \quad ...(11)$$

Solving (10) and (11) using an iterative technique, we get a^2=5.0514 and c=9.37916.

Here, we may point out this SOSRD has only 57 design points for 6-factors, whereas the corresponding SOSRDs obtained by Victorbabu and Narasimham [8], [10] using the BIB design (v=6, b=15, r=5, k=2, λ=1) and pairwise balanced design (v=6, b=7, r=3, k_1=3, k_2=2, λ=1) need 73 and 69 design points respectively.

Example (ii) : The design points,

$$[1-(v = 8,\ b_1 = 8,\ r_1 = 3,\ k_1 = 3,\ \lambda_1 = 1,\ \lambda_2 = 0)]\ 2^3 \cup$$

$$[a_1-(v = 8,\ b_2 = 4,\ r_2 = 1,\ k_2 = 2,\ \lambda_1' = 0,\ \lambda_2' = 1)]\ 2^2 \cup$$

$$(a, 0, \ldots, 0)\ 2^1 \cup (n_0 = 1)$$

will give a SOSRD in N=97 design points with a=2.1811 and c=9.657731.

For $v = 8$ factors, this new method needs 97 design points, whereas the corresponding SOSRD constructed through BIB design ($v = 8$, $b = 28$, $r = 7$, $k = 2$, $\lambda = 1$) and pairwise balanced design ($v = 8$, $b = 15$, $r = 6$, $k_1 = 4$, $k_2 = 3$, $k_3 = 2$, $\lambda = 2$) need 129 and 257 design points respectively.

Example (iii) : The design points,

$$[1-(v = 10,\ b_1 = 8,\ r_1 = 4,\ k_1 = 5,\ \lambda_1 = 2,\ \lambda_2 = 0)]\ 2^4 \cup$$

$$[a_1-(v = 10,\ b_2 = 5,\ r_2 = 1,\ k_2 = 2,\ \lambda_1' = 0,\ \lambda_2' = 1)]\ 2^2 \cup$$

$$(a, 0, \ldots, 0)\ 2^1 \cup (n_o = 1)$$

will give a SOSRD in N=169 design points with a=2.9568 and c=7.777138.

In the case of 10-factors, this new method needs 169 design points, whereas the corresponding SOSRD constructed through BIB design ($v = 10$, $b = 45$, $r = 9$, $k = 2$, $\lambda = 1$) and pairwise balanced designs ($v = 10$, $b = 11$, $r = 5$, $k_1 = 5$, $k_2 = 4$, $\lambda = 2$) need 201 and 197 design points respectively.

We note thus the new method sometimes leads to designs with less number of design points.

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